Partial Differential Equations: Final Exam

MartiniPlaza, Monday 10 April 2017, 09:00–12:00
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of your exam sheet and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explain why the conditions for using such results are satisfied.
- 10 points are "free". There are 5 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (20 points)

Separate the variables for the equation

$$\frac{2}{t}u_t = u_{xx} + u, \quad 0 < x < \pi, \quad t > 1,$$

with the boundary conditions $u(0,t) = u(\pi,t) = 0$, and find the general solution $u(x,t)$ in the form of a series. Given the initial condition $u(x,1) = 1$, find the asymptotic form of the solution as $t \to \infty$.

Solution

Let

$$u(x,t) = X(x)T(t).$$

Then

$$\frac{2}{t}XT' = X''T + XT,$$

giving

$$\frac{2T'}{tT} - 1 = \frac{X''}{X} = -\lambda.$$

The boundary conditions give $X(0) = X(\pi) = 0$ and therefore

$$\lambda_n = n^2, \quad X_n = \sin(nx), \quad n = 1, 2, 3, \ldots.$$

The corresponding t equation is

$$T_n' = \frac{1}{2}(1-n^2)tT_n,$$

giving

$$\frac{dT_n}{T_n} = \frac{1}{2}(1-n^2)t\,dt \Rightarrow \log|T_n| = \frac{1}{4}(1-n^2)t^2 + C \Rightarrow T_n = Ae^{\frac{1}{4}(1-n^2)t^2}.$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{\frac{1}{4}(1-n^2)t^2} \sin(nx).$$

As $t \to \infty$ we have

$$\lim_{t\to\infty} u(x,t) = A_1 \sin(x).$$

At $t = 1$ we have

$$u(x,1) = \sum_{n=1}^{\infty} A_n e^{\frac{1}{4}(1-n^2)} \sin(nx) = 1.$$

The coefficient A_1 is given by

$$A_1 = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin x \, dx = \frac{4}{\pi}.$$

Therefore

$$\lim_{t\to\infty} u(x,t) = \frac{4}{\pi} \sin(x).$$

Question 2 (20 points)

Solve the partial differential equation

$$u_x + x u_y = 1,$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with initial data $u(0,y) = f(y)$.

Solution

Solution 1

The characteristic curves are given by

$$\frac{dy}{dx} = x \Rightarrow y = \tfrac{1}{2}x^2 + C \Rightarrow C = y - \tfrac{1}{2}x^2.$$

Then we notice that along a characteristic curve we have

$$\frac{du}{dx} = u_x + u_y \frac{dy}{dx} = u_x + x u_y = 1.$$

Therefore, if (x,y) belongs on the characteristic curve that goes through the point $(0, y_0)$, we have

$$u(x,y) = u(0,y_0) + x.$$

Since the two points belong on the same characteristic curve we have

$$C = y - \tfrac{1}{2}x^2 = y_0,$$

and, using the initial data,

$$u(0, y_0) = f(y_0) = f(y - \tfrac{1}{2}x^2).$$

Therefore,

$$u(x, y) = f(y - \tfrac{1}{2}x^2) + x.$$

Solution 2

Yet another way to solve this problem is to notice that $1 = (x)_x$, so we can write the equation in the form

$$(u - x)_x + xu_y = 0 \Rightarrow (u - x)_x + x(u - x)_y = 0.$$

Let $v = u - x$. Then we have

$$v_x + xv_y = 0.$$

The characteristic curves are given by solving

$$\frac{dy}{dx} = x \Rightarrow y = \frac{1}{2}x^2 + C \Rightarrow C = y - \frac{1}{2}x^2.$$

Therefore,

$$v = g(y - \tfrac{1}{2}x^2) \Rightarrow u = g(y - \tfrac{1}{2}x^2) + x.$$

Using the initial data we find

$$u(0, y) = g(y) = f(y).$$

Thus $g = f$ and the solution is

$$u = x + f(y - \tfrac{1}{2}x^2).$$

Solution 3

Alternatively, after finding the characteristic curves $C = y - \frac{1}{2}x^2$ we can consider the coordinate transformation $\eta = y - \frac{1}{2}x^2$, $\xi = x$. Then

$$u_x = -xu_\eta + u_\xi, \quad u_y = u_\eta.$$

Then

$$u_x + xu_y = -xu_\eta + u_\xi + xu_\eta = u_\xi = 1.$$

Therefore,

$$u = \xi + g(\eta) = x + g(y - \tfrac{1}{2}x^2).$$

For $x = 0$ we find $u(0, y) = f(y) = g(y)$. Thus $g = f$ and the solution is

$$u = x + f(y - \tfrac{1}{2}x^2).$$

Question 3 (20 points)

Find the solution to the problem $u_{tt} = u_{xx} + \cos(\omega t)$, where $0 < x < \pi$, with the boundary conditions $u(t,0) = u(t,\pi) = 0$, and initial data $u(0,x) = u_t(0,x) = 0$. Here, $\omega > 0$, and ω is *not* an integer.

Solution

Write

$$u = \sum_{n=1}^{\infty} u_n(t)\sin(nx), \quad u_{tt} = \sum_{n=1}^{\infty} v_n(t)\sin(nx), \quad u_{xx} = \sum_{n=1}^{\infty} w_n(t)\sin(nx).$$

Then we have

$$\begin{aligned} w_n &= \frac{2}{\pi}\int_0^{\pi} u_{xx}\sin(nx)dx \\ &= \frac{2}{\pi}\left([u_x \sin(nx)]_{x=0}^{x=\pi} - n\int_0^{\pi} u_x \cos(nx)dx \right) \\ &= -\frac{2n}{\pi}\int_0^{\pi} u_x\cos(nx)dx \\ &= -\frac{2n}{\pi}\left([u\cos(nx)]_{x=0}^{x=\pi} + n\int_0^{\pi} u\sin(nx)dx \right) \\ &= -\frac{2n^2}{\pi}\int_0^{\pi} u\sin(nx)dx \\ &= -n^2 u_n, \end{aligned}$$

and

$$\begin{aligned} v_n &= \frac{2}{\pi}\int_0^{\pi} u_{tt}\sin(nx)dx \\ &= \frac{d^2}{dt^2}\frac{2}{\pi}\int_0^{\pi} u\sin(nx)dx \\ &= u_n''. \end{aligned}$$

We obtain the equation

$$\begin{aligned} u_n'' + n^2 u_n &= \frac{2}{\pi}\int_0^{\pi} (u_{tt} - u_{xx})\sin(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi} \cos(\omega t)\sin(nx)dx \\ &= \cos(\omega t)\frac{2}{\pi}\int_0^{\pi}\sin(nx)dx \\ &= d_n\cos(\omega t), \end{aligned}$$

where

$$d_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx,$$

are the Fourier sine coefficients of 1. We compute

$$d_n = \frac{2}{\pi}\frac{1-(-1)^n}{n}.$$

The solution of the ODE for u_n is

$$u_n = A\cos(nt) + B\sin(nt) + d_n \frac{\cos(\omega t)}{n^2 - \omega^2}.$$

For $t = 0$ we have $u_n = 0$ and $u_n' = 0$ therefore

$$A = -d_n \frac{1}{n^2 - \omega^2}, \quad B = 0,$$

and

$$u_n = d_n \frac{\cos(\omega t) - \cos(nt)}{n^2 - \omega^2}.$$

So the solution is

$$u(x,t) = \sum_{n=1}^{\infty} d_n \frac{\cos(\omega t) - \cos(nt)}{n^2 - \omega^2} \sin(nx).$$

Question 4 (15 points)

Compute the Fourier coefficients for the complex Fourier series of the function $f(x) = x$, $-\pi < x < \pi$. Discuss the L^2, uniform, and pointwise convergence of the series, giving sufficient justification of your statements. Specifically, does the series converge in the L^2 sense, uniformly, and pointwise? If yes, how did you conclude this and what is the limit? If no, how did you conclude this?

Solution

The complex Fourier series for a function $f(x)$ of period 2π is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

We compute, for $n \neq 0$,

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx \\
&= \frac{i}{2\pi n} \left(\left[x e^{-inx} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-inx}\, dx \right) \\
&= \frac{i}{2\pi n} \left(\pi (e^{-in\pi} + e^{in\pi}) - \frac{i}{n} (e^{-in\pi} - e^{in\pi}) \right) \\
&= \frac{i}{2\pi n} \left(2\pi \cos n\pi - \frac{2}{n} \sin n\pi \right) \\
&= \frac{i}{n} \cos n\pi \\
&= \frac{i}{n} (-1)^n.
\end{aligned}$$

For $n = 0$, we have

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0.$$

The Fourier series converges to $f(x)$ in the L^2 sense, since

$$\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3} < \infty.$$

For pointwise convergence, we notice that the periodic extension $f_{\text{per}}(x)$ of $f(x)$ is piecewise continuous and is discontinuous only for $x = (2k+1)\pi$, $k \in \mathbb{Z}$. This means that the series converges pointwise to $f_{\text{per}}(x)$ for all x outside the set of discontinuities and to 0 at the discontinuities. The latter follows from $f(\pi^-) = \pi$, $f(\pi^+) = -\pi$, so $\frac{1}{2}[f(\pi^-) + f(\pi^+)] = 0$, and from 2π-periodicity.

Finally, for uniform convergence, we know that the function does not satisfy the requirements of the corresponding theorem since its extension is not continuous (corresponding to the fact that the function does not satisfy the right boundary conditions, that is, periodic boundary conditions since we have a complex Fourier series). Moreover, when the function has a jump discontinuity at a point then the Gibbs phenomenon implies that the series does not converge uniformly.

Question 5 (15 points)

Consider the diffusion equation $u_t = ku_{xx}$, $k > 0$, where $0 < x < \ell$ and $t > 0$. Consider two solutions, $u(x,t)$ and $v(x,t)$, of the given equation, such that $u(x,0) \leq v(x,0)$ for all $0 < x < \ell$, and $u(0,t) \leq v(0,t)$, $u(\ell,t) \leq v(\ell,t)$, for all $t > 0$. Prove that $u(x,t) \leq v(x,t)$ for $0 \leq t < \infty$, $0 \leq x \leq \ell$.

Solution

Let

$$w(x,t) = u(x,t) - v(x,t).$$

Then w solves the diffusion equation and it satisfies $w(x,0) \leq 0$, $w(0,t) \leq 0$, and $w(\ell,t) \leq 0$.

From the maximum principle we know that w attains its maximum value M at one of the sides $t = 0$, $x = 0$, or $x = \ell$. Since $w \leq 0$ in all of these sides we conclude that $M \leq 0$. Therefore, from the maximum principle, $w(x,t) \leq M \leq 0$ for $0 \leq t < \infty$, $0 \leq x \leq \ell$.

We conclude that

$$u(x,t) \leq v(x,t),$$

for $0 \leq t < \infty$, $0 \leq x \leq \ell$.

Facts and formulas

The eigenvalues and eigenfunction for the problem $X'' = -\lambda X$, $X(0) = X(\ell) = 0$, are given by

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right),$$

for $n = 1, 2, 3, \ldots$.

The eigenvalues and eigenfunction for the problem $X'' = -\lambda X$, $X'(0) = X'(\ell) = 0$, are given by

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right),$$

for $n = 1, 2, 3, \ldots$, together with

$$\lambda_0 = 0, \quad X_0(x) = \frac{1}{2}.$$